Kuwait University | Math 102 | May 14, 2009
Dept. of Math. and Comp. Science | Second Exam | Duration: 90 minutes

Calculators and mobile phones are not allowed.
Answer all of the following questions.

1. Evaluate the following. [1+3 pts.]

(a) $\lim_{x\to\infty} \frac{\tan^{-1} x}{1+x^2}$

(b) $\lim_{x\to\infty} \left(\frac{x+2}{x+3}\right)^{2x-1}$

2. Evaluate the following. [3.5 pts. each]

(a) $\int \frac{x^2}{(9-x^2)^{3/2}}\,dx$

(b) $\int (\sec^6 x)\sqrt{\tan x}\,dx$

(c) $\int \frac{\sin 2x}{\sqrt{5-\cos x}}\,dx$

(d) $\int \frac{2-3x^2}{(2x+1)(x^2+2x+2)}\,dx$

(e) $\int \frac{dx}{(1+\cos x)\sin x}$

(f) $\int x(\ln x + e^x)\,dx$

1. (a) Since $\lim_{x\to\infty} \tan^{-1} x = \pi/2$ and $\lim_{x\to\infty} 1+x^2 = \infty$, it follows that $\lim_{x\to\infty} \dfrac{\tan^{-1} x}{1+x^2} = 0$.

(b) $\lim_{x\to\infty} \ln(f(x)) = \lim_{x\to\infty} (2x-1)\ln\left(\dfrac{x+2}{x+3}\right) = \lim_{x\to\infty} \dfrac{\ln[(x+2)/(x+3)]}{(2x-1)^{-1}}$ which has the indeterminate form 0/0. Applying L'Hospital's Rule and simplifying yields
$\lim_{x\to\infty} \ln(f(x)) = \lim_{x\to\infty} \dfrac{-1}{2}\dfrac{(2x-1)^2}{(x+2)(x+3)} = -2$. Therefore, $\lim_{x\to\infty} f(x) = e^{-2}$.

2. (a) The substitution $x = 3\sin\theta$, $dx = 3\cos\theta\, d\theta$, yields

$$\int \frac{x^2}{(9-x^2)^{3/2}}\, dx = \int \tan^2\theta\, d\theta = \tan\theta - \theta + C = \frac{x}{\sqrt{9-x^2}} - \sin^{-1}\frac{x}{3} + C$$

(b) The substitution $u = \tan x$, $du = \sec^2 x\, dx$ yields

$$\int (\sec^6 x)\sqrt{\tan x}\, dx = \int (u^2+1)^2\sqrt{u}\, du = \frac{2}{11}u^{11/2} + \frac{4}{7}u^{7/2} + \frac{2}{3}u^{3/2} + C,$$

where $u = \tan x$.

(c) The substitution $u^2 = 5 - \cos x$, $2u\, du = \sin x\, dx$, yields

$$\int \frac{\sin 2x}{\sqrt{5-\cos x}}\, dx = 4\int (5-u^2)\, du = 20u - \frac{4}{3}u^3 + C,$$

where $u = \sqrt{5-\cos x}$.

(d) The partial fraction decomposition takes the form

$$\frac{2-3x^2}{(2x+1)(x^2+2x+2)} = \frac{A}{2x+1} + \frac{Bx+C}{x^2+2x+2},$$

where the constants are $A = 1$, $B = -2$ and $C = 0$. Therefore,

$$\int \frac{2-3x^2}{(2x+1)(x^2+2x+2)}\, dx = \frac{1}{2}\ln|2x+1| - \ln(x^2+2x+2) + 2\tan^{-1}(x+1) + C$$

(e) The substitution $u = \tan\frac{x}{2}$, yields

$$\int \frac{dx}{(1+\cos x)\sin x} = \int \frac{1+u^2}{2u}\, du = \frac{1}{2}\ln|u| + \frac{1}{4}u^2 + C,$$

where $u = \tan\frac{x}{2}$.

(f) We write $\int x(\ln x + e^x)\, dx = \int x\ln x\, dx + \int xe^x\, dx$ and apply integration by parts to both integrals to obtain $\int x\ln x\, dx = \dfrac{x^2}{2}\ln x - \dfrac{1}{2}\int x\, dx$ and $\int xe^x\, dx = xe^x - \int e^x\, dx$. The final answer is then

$$\int x(\ln x + e^x)\, dx = \frac{x^2}{2}\ln x - \frac{1}{4}x^2 + xe^x - e^x + C.$$